64. **Sports** A snowmaking machine at a ski resort can produce enough snow for a beginner's ski trail in 16 hours. With a typical natural snowfall, it takes 24 hours to deposit enough snow to open the beginner's ski trail. If the snowmaking machine is run during a typical natural snowfall, how long will it take to deposit enough snow to open the beginner's trail?

65. **Road Construction** A new machine that deposits cement for a road requires 12 hours to complete a one-half mile section of road. An older machine requires 16 hours to pave the same amount of road. After depositing cement for 4 hours, the new machine develops a mechanical problem and quits working. The older machine is brought into place and continues the job. How long does it take the older machine to complete the job?

66. **Masonry** A mason can lay the bricks in a sidewalk in 12 hours. The mason's apprentice requires 16 hours to do the same job. After working together for 4 hours, the mason leaves for another job, and the apprentice continues working. How long will it take the apprentice to complete the job?

Enrichment Exercises

67. **Chemistry** A large vat contains a solution that is 30% salt and 70% water. Eight kilograms of the solution are removed from the vat and placed in an open container. After 2 kg of water evaporate from the open container, a lab technician takes an additional 2 kilograms of solution from the vat and places it in the open container. Now what is the percent salt concentration in the open container?

68. **Chemistry** A 10-pound solution is 99% water and 1% sugar. After some of the water evaporates, the solution is 98% water. Now what is the weight of the solution?

69. **Uniform Motion** Marlene rides her bicycle to her friend Jon's house and returns home by the same route. Marlene rides her bike at constant speeds of 6 mph on level ground, 4 mph when going uphill, and 12 mph when going downhill. If her total time riding was 1 hour, how far is it to Jon's house? (*Hint*: Let d_1 be the distance traveled on level ground and let d_2 be the distance traveled on the hill. Then the distance between the two houses is $d_1 + d_2$. Write an equation for the total time. For instance, the time spent in traveling to Jon's house on level ground is $\frac{d_1}{6}$.)

SECTION 1.3

- Solving Quadratic Equations by Factoring
- Solving Quadratic Equations by Taking Square Roots
- Solving Quadratic Equations by Completing the Square
- Solving Quadratic Equations by Using the Quadratic Formula
- The Discriminant of a Quadratic Equation
- Applications of Quadratic Equations

Quadratic Equations

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A5.

PS1. Factor: $x^2 - x - 42$ [P.4]

PS2. Factor: $6x^2 - x - 15$ [P.4]

PS3. Write $3 + \sqrt{-16}$ in $a + bi$ form. [P.6]

PS4. If $a = -3$, $b = -2$, and $c = 5$, evaluate $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ [P.1/P.2]

PS5. If $a = 2$, $b = -3$, and $c = 1$, evaluate $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ [P.1/P.2]

PS6. If $x = 3 - i$, evaluate $x^2 - 6x + 10$. [P.6]

Solving Quadratic Equations by Factoring

In Section 1.1 you solved linear equations. In this section you will learn to solve a type of equation that is referred to as a *quadratic equation*.

Math Matters

The term *quadratic* is derived from the Latin word *quadrare*, which means "to make square." Because the area of a square that measures x units on each side is x^2, we refer to equations that can be written in the form $ax^2 + bx + c = 0$ as equations that are "quadratic in x."

Definition of a Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the **standard quadratic form**

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

Several methods can be used to solve a quadratic equation. For instance, if you can factor $ax^2 + bx + c$ into linear factors, then $ax^2 + bx + c = 0$ can be solved by applying the following property.

The Zero Product Principle

If A and B are algebraic expressions such that $AB = 0$, then $A = 0$ or $B = 0$.

The zero product principle states that if the product of two factors is 0, then at least one of the factors must be 0. In Example 1, the zero product principle is used to solve a quadratic equation.

EXAMPLE 1 Solve by Factoring

Solve each quadratic equation by factoring.

a. $x^2 + 2x - 15 = 0$ **b.** $2x^2 - 5x = 12$

Solution

a. $x^2 + 2x - 15 = 0$

$(x - 3)(x + 5) = 0$ • Factor.

$x - 3 = 0$ or $x + 5 = 0$ • Set each factor equal to 0.

$x = 3$ $x = -5$ • Solve each linear equation.

A check shows that 3 and -5 are both solutions of $x^2 + 2x - 15 = 0$.

b. $2x^2 - 5x = 12$

$2x^2 - 5x - 12 = 0$ • Write in standard quadratic form.

$(x - 4)(2x + 3) = 0$ • Factor.

$x - 4 = 0$ or $2x + 3 = 0$ • Set each factor equal to 0.

$x = 4$ $2x = -3$ • Solve each linear equation.

$x = -\frac{3}{2}$

A check shows that 4 and $-\frac{3}{2}$ are both solutions of $2x^2 - 5x = 12$.

▶ **Try Exercise 12, page 105**

Some quadratic equations have a solution that is called a *double root*. For instance, consider $x^2 - 8x + 16 = 0$. Solving this equation by factoring, we have

$x^2 - 8x + 16 = 0$

$(x - 4)(x - 4) = 0$ • Factor.

$x - 4 = 0$ or $x - 4 = 0$ • Set each factor equal to 0.

$x = 4$ $x = 4$ • Solve each linear equation.

The only solution of $x^2 - 8x + 16 = 0$ is 4. In this situation, the single solution 4 is called a **double solution** or **double root** because it was produced by solving the two identical equations $x - 4 = 0$, both of which have 4 as a solution.

Solving Quadratic Equations by Taking Square Roots

Recall that $\sqrt{x^2} = |x|$. This principle can be used to solve some quadratic equations by taking the square root of each side of the equation.

In the following example, we use this idea to solve $x^2 = 25$.

$$x^2 = 25$$

$$\sqrt{x^2} = \sqrt{25}$$ • Take the square root of each side.

$$|x| = 5$$ • Use the fact that $\sqrt{x^2} = |x|$ and $\sqrt{25} = 5$.

$$x = -5 \quad \text{or} \quad x = 5$$ • Solve the absolute value equation.

The solutions are -5 and 5.

TO REVIEW

Square Roots of Variable Expressions
See page 23.
Absolute Value Equations
See page 79.

We will refer to the preceding method of solving a quadratic equation as the **square root procedure.**

The Square Root Procedure

If $x^2 = c$, then $x = \sqrt{c}$ or $x = -\sqrt{c}$, which can also be written as $x = \pm\sqrt{c}$.

EXAMPLE

If $x^2 = 9$, then $x = \sqrt{9} = 3$ or $x = -\sqrt{9} = -3$. This can be written as $x = \pm 3$.

If $x^2 = 7$, then $x = \sqrt{7}$ or $x = -\sqrt{7}$. This can be written as $x = \pm\sqrt{7}$.

If $x^2 = -4$, then $x = \sqrt{-4} = 2i$ or $x = -\sqrt{-4} = -2i$. This can be written as $x = \pm 2i$.

EXAMPLE 2 Solve by Using the Square Root Procedure

Use the square root procedure to solve each equation.

a. $3x^2 + 12 = 0$ **b.** $(x + 1)^2 = 48$

Solution

a. $3x^2 + 12 = 0$

$$3x^2 = -12$$ • Solve for x^2.

$$x^2 = -4$$

$$x = \pm\sqrt{-4}$$ • Take the square root of each side of the equation and insert a plus-or-minus sign in front of the radical.

$$x = -2i \quad \text{or} \quad x = 2i$$

The solutions are $-2i$ and $2i$.

b. $(x + 1)^2 = 48$

$$x + 1 = \pm\sqrt{48}$$ • Take the square root of each side of the equation and insert a plus-or-minus sign in front of the radical.

$$x + 1 = \pm 4\sqrt{3}$$ • Simplify.

$$x = -1 \pm 4\sqrt{3}$$

$$x = -1 - 4\sqrt{3} \quad \text{or} \quad x = -1 + 4\sqrt{3}$$

The solutions are $-1 - 4\sqrt{3}$ and $-1 + 4\sqrt{3}$.

Try Exercise 30, page 105

Solving Quadratic Equations by Completing the Square

Consider two binomial squares and their perfect-square trinomial products.

Square of a Binomial	=	Perfect-Square Trinomial
$(x+5)^2$	=	$x^2+10x+25$
$(x-3)^2$	=	x^2-6x+9

Math Matters

Mathematicians have studied quadratic equations for centuries. Many of the initial quadratic equations they studied resulted from attempts to solve geometry problems. One of the most famous, which dates from around 500 B.C., concerns "squaring a circle." The question was, Is it possible to construct a square whose area is the same as the area of a given circle? For these early mathematicians, *to construct* meant to draw with only a straightedge and a compass. It was approximately 2300 years later when mathematicians proved that such a construction is impossible.

In each of the preceding perfect-square trinomials, the coefficient of x^2 is 1 and the constant term is the square of half the coefficient of the x term.

$$x^2+10x+25, \quad \left(\frac{1}{2}\cdot 10\right)^2 = 25$$

$$x^2-6x+9, \quad \left(\frac{1}{2}\cdot(-6)\right)^2 = 9$$

Adding to a binomial of the form x^2+bx the constant term that makes the binomial a perfect-square trinomial is called **completing the square**. For example, to complete the square of x^2+8x, add

$$\left(\frac{1}{2}\cdot 8\right)^2 = 16$$

to produce the perfect-square trinomial $x^2+8x+16$.

Completing the square is a powerful procedure that can be used to solve *any* quadratic equation. For instance, to solve $x^2-6x+13=0$, first isolate the variable terms on one side of the equation and the constant term on the other side.

$$x^2-6x=-13$$ • Subtract 13 from each side of the equation.

$$x^2-6x+9=-13+9$$ • Complete the square by adding $\left[\frac{1}{2}(-6)\right]^2=9$ to each side of the equation.

$$(x-3)^2=-4$$ • Factor and solve by the square root procedure.

$$x-3=\pm\sqrt{-4}$$

$$x-3=\pm 2i$$

$$x=3\pm 2i$$

The solutions of $x^2-6x+13=0$ are $3-2i$ and $3+2i$. You can check these solutions by substituting each solution into the original equation. For instance, the following check shows that $3-2i$ does satisfy the original equation.

$$x^2-6x+13=0$$

$$(3-2i)^2-6(3-2i)+13 \stackrel{?}{=} 0$$ • Substitute $3-2i$ for x.

$$9-12i+4i^2-18+12i+13 \stackrel{?}{=} 0$$ • Simplify.

$$4+4(-1) \stackrel{?}{=} 0$$

$$0=0$$ • The left side equals the right side, so $3-2i$ checks.

EXAMPLE 3 Solve by Completing the Square

Solve $x^2=2x+6$ by completing the square.

Solution

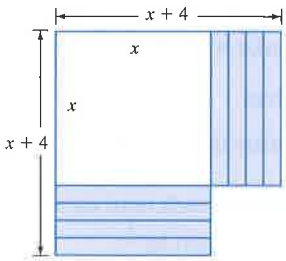
$$x^2=2x+6$$

$$x^2-2x=6$$ • Isolate the constant term.

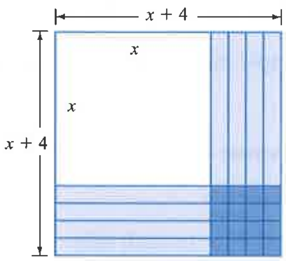
$$x^2-2x+1=6+1$$ • Complete the square.

Math Matters

Ancient mathematicians thought of "completing the square" in a geometric manner. For instance, to complete the square of $x^2 + 8x$, draw a square that measures x units on each side and add four rectangles that measure 1 unit by x units to the right side of and the bottom of the square.



Each of the rectangles has an area of x square units, so the total area of the figure is $x^2 + 8x$. To make this figure a complete square, we must add 16 squares that measure 1 unit by 1 unit, as shown below.



This figure is a *complete square* whose area is

$$(x + 4)^2 = x^2 + 8x + 16$$

$$(x - 1)^2 = 7 \quad \text{• Factor and simplify.}$$

$$x - 1 = \pm\sqrt{7} \quad \text{• Apply the square root procedure.}$$

$$x = 1 \pm \sqrt{7} \quad \text{• Solve for } x.$$

The exact solutions of $x^2 = 2x + 6$ are $1 - \sqrt{7}$ and $1 + \sqrt{7}$. A calculator can be used to show that $1 - \sqrt{7} \approx -1.646$ and $1 + \sqrt{7} \approx 3.646$. The decimals -1.646 and 3.646 are approximate solutions of $x^2 = 2x + 6$.

▶ **Try Exercise 46, page 105**

Completing the square by adding the square of half the coefficient of the x term requires that the coefficient of the x^2 term be 1. If the coefficient of the x^2 term is not 1, then first multiply each term on each side of the equation by the reciprocal of the coefficient of x^2 to produce a coefficient of 1 for the x^2 term.

EXAMPLE 4 Solve by Completing the Square

Solve $2x^2 + 8x - 1 = 0$ by completing the square.

Solution

$$2x^2 + 8x - 1 = 0$$

$$2x^2 + 8x = 1 \quad \text{• Isolate the constant term.}$$

$$\frac{1}{2}(2x^2 + 8x) = \frac{1}{2}(1) \quad \text{• Multiply both sides of the equation by the reciprocal of the coefficient of } x^2.$$

$$x^2 + 4x = \frac{1}{2}$$

$$x^2 + 4x + 4 = \frac{1}{2} + 4 \quad \text{• Complete the square.}$$

$$(x + 2)^2 = \frac{9}{2} \quad \text{• Factor and simplify.}$$

$$x + 2 = \pm\sqrt{\frac{9}{2}} \quad \text{• Apply the square root procedure.}$$

$$x = -2 \pm 3\sqrt{\frac{1}{2}} \quad \text{• Solve for } x.$$

$$x = -2 \pm 3\frac{\sqrt{2}}{2} \quad \text{• Simplify.}$$

$$x = \frac{-4 \pm 3\sqrt{2}}{2}$$

The solutions are $\frac{-4 - 3\sqrt{2}}{2}$ and $\frac{-4 + 3\sqrt{2}}{2}$.

▶ **Try Exercise 44, page 105**

Solving Quadratic Equations by Using the Quadratic Formula

Completing the square for $ax^2 + bx + c = 0$ $(a \neq 0)$ produces a formula for x in terms of the coefficients a, b, and c. The formula is known as the *quadratic formula*, and it can be used to solve *any* quadratic equation.

Math Matters

Evariste Galois (1811–1832) The quadratic formula provides the solutions to the general quadratic equation

$$ax^2 + bx + c = 0$$

Formulas also have been developed to solve the general cubic

$$ax^3 + bx^2 + cx + d = 0$$

and the general quartic

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

However, the French mathematician Evariste Galois, shown above, proved that there are no formulas that can be used to solve "by radicals" general equations of degree 5 or larger.

Shortly after completion of his remarkable proof, Galois was shot in a duel. It has been reported that as Galois lay dying, he asked his brother, Alfred, to "Take care of my work. Make it known. Important." When Alfred broke into tears, Evariste said, "Don't cry, Alfred. I need all my courage to die at twenty."

(*Source: Whom the Gods Love*, by Leopold Infeld, The National Council of Teachers of Mathematics, 1978, p. 299.)

The Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof

We assume a is a positive real number. If a were a negative real number, then we could multiply each side of the equation by -1 to make it positive.

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$ • Given.

$$ax^2 + bx = -c$$ • Isolate the constant term.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$ • Multiply each term on each side of the equation by $\frac{1}{a}$.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$ • Complete the square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$ • Factor the left side. Simplify the powers on the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4a}{4a} \cdot \frac{c}{a}$$ • Use a common denominator to simplify the right side.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$ • Apply the square root procedure.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$ • Because $a > 0, \sqrt{4a^2} = 2a$.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ • Add $-\frac{b}{2a}$ to each side.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ ■

As a general rule, you should first try to solve quadratic equations by factoring. If the factoring process proves difficult, then solve by using the quadratic formula.

EXAMPLE 5 Solve by Using the Quadratic Formula

Use the quadratic formula to solve each of the following.

a. $x^2 = 3x + 5$ **b.** $4x^2 - 4x + 3 = 0$

Solution

a. $$x^2 = 3x + 5$$

$$x^2 - 3x - 5 = 0$$ • Write the equation in standard form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ • Use the quadratic formula.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)}$$ • $a = 1, b = -3, c = -5$.

$$= \frac{3 \pm \sqrt{29}}{2}$$

$$x = \frac{3 - \sqrt{29}}{2} \quad \text{or} \quad \frac{3 + \sqrt{29}}{2}$$

The solutions are $\frac{3 - \sqrt{29}}{2}$ and $\frac{3 + \sqrt{29}}{2}$.

b. $4x^2 - 4x + 3 = 0$ • The equation is in standard form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ • Use the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(3)}}{2(4)}$$ • $a = 4, b = -4, c = 3$.

$$= \frac{4 \pm \sqrt{16 - 48}}{2(4)} = \frac{4 \pm \sqrt{-32}}{8}$$

$$= \frac{4 \pm 4i\sqrt{2}}{8}$$

$$x = \frac{4 - 4i\sqrt{2}}{8} = \frac{1}{2} - \frac{\sqrt{2}}{2}i \quad \text{or} \quad x = \frac{4 + 4i\sqrt{2}}{8} = \frac{1}{2} + \frac{\sqrt{2}}{2}i$$

The solutions are $\frac{1}{2} - \frac{\sqrt{2}}{2}i$ and $\frac{1}{2} + \frac{\sqrt{2}}{2}i$.

Try Exercise 58, page 105

Question • Can the quadratic formula be used to solve any quadratic equation $ax^2 + bx + c = 0$ with real coefficients and $a \neq 0$?

The Discriminant of a Quadratic Equation

The solutions of $ax^2 + bx + c = 0$, $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the radical, $b^2 - 4ac$, is called the **discriminant** of the equation $ax^2 + bx + c = 0$. If $b^2 - 4ac \geq 0$, then $\sqrt{b^2 - 4ac}$ is a real number. If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number. Thus the sign of the discriminant can be used to determine whether the solutions of a quadratic equation are real numbers.

TO REVIEW

Complex Conjugates
See page 63.

The Discriminant and the Solutions of a Quadratic Equation

The equation $ax^2 + bx + c = 0$, with real coefficients and $a \neq 0$, has as its discriminant $b^2 - 4ac$.

- If $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ has *two distinct real solutions.*
- If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has *one real solution.* The solution is a double solution.
- If $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has *two distinct nonreal complex solutions.* The solutions are conjugates of each other.

Answer • Yes. However, it is sometimes easier to find the solutions by factoring, by the square root procedure, or by completing the square.

EXAMPLE 6 Use the Discriminant to Determine the Number of Real Solutions

For each equation, determine the discriminant and state the number of real solutions.

a. $2x^2 - 5x + 1 = 0$

b. $3x^2 + 6x + 7 = 0$

c. $x^2 + 6x + 9 = 0$

Solution

a. The discriminant of $2x^2 - 5x + 1 = 0$ is $b^2 - 4ac = (-5)^2 - 4(2)(1) = 17$. Because the discriminant is positive, $2x^2 - 5x + 1 = 0$ has two distinct real solutions.

b. The discriminant of $3x^2 + 6x + 7 = 0$ is $b^2 - 4ac = 6^2 - 4(3)(7) = -48$. Because the discriminant is negative, $3x^2 + 6x + 7 = 0$ has no real solutions.

c. The discriminant of $x^2 + 6x + 9 = 0$ is $b^2 - 4ac = 6^2 - 4(1)(9) = 0$. Because the discriminant is 0, $x^2 + 6x + 9 = 0$ has one real solution.

▶ Try Exercise 72, page 105

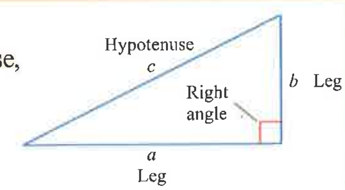
Applications of Quadratic Equations

A **right triangle** contains one 90° angle. The side opposite of the 90° angle is called the **hypotenuse.** The other two sides are called **legs.** The lengths of the sides of a right triangle are related by a theorem known as the Pythagorean Theorem.

The Pythagorean Theorem states that the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs. This theorem is often used to solve applications that involve right triangles.

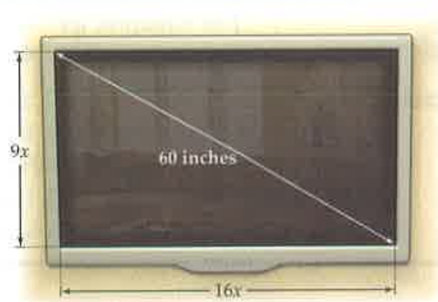
The Pythagorean Theorem

If a and b denote the lengths of the legs of a right triangle and c the length of the hypotenuse, then $c^2 = a^2 + b^2$.



EXAMPLE 7 Determine the Dimensions of a Television Screen

A television screen measures 60 inches diagonally, and its *aspect ratio* is 16 to 9. This means that the ratio of the width of the screen to the height of the screen is 16 to 9. Find the width and height of the screen.



A 60-inch television screen with a 16:9 aspect ratio.

Note

Many movies are designed to be shown on a screen that has a 16-to-9 aspect ratio.

Solution

Let $16x$ represent the width of the screen and let $9x$ represent the height of the screen. Applying the Pythagorean Theorem gives us

$$(16x)^2 + (9x)^2 = 60^2$$
$$256x^2 + 81x^2 = 3600 \quad \text{• Solve for } x.$$
$$337x^2 = 3600$$
$$x^2 = \frac{3600}{337}$$
$$x = \sqrt{\frac{3600}{337}} \approx 3.268 \text{ inches}$$

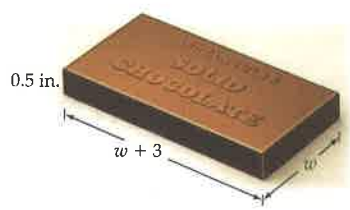
• Apply the square root procedure. The plus-or-minus sign is not used in this application because we know x is positive.

The height of the screen is about $9(3.268) \approx 29.4$ inches, and the width of the screen is about $16(3.268) \approx 52.3$ inches.

▸ Try Exercise 82, page 106

EXAMPLE 8 Determine the Dimensions of a Candy Bar

A company makes rectangular solid candy bars that measure 5 inches by 2 inches by 0.5 inch. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should the dimensions be for the new candy bar if the company keeps the height at 0.5 inch and makes the length of the candy bar 3 inches longer than the width?



Solution

The volume of a rectangular solid is given by $V = lwh$. The original candy bar had a volume of $5 \cdot 2 \cdot 0.5 = 5$ cubic inches. The new candy bar will have a volume of $80\%(5) = 0.80(5) = 4$ cubic inches.

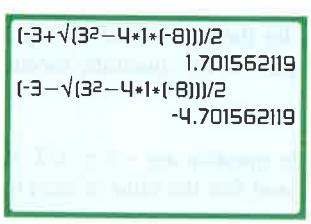
Let w represent the width and $w + 3$ represent the length of the new candy bar. For the new candy bar we have

$$lwh = V$$
$$(w+3)(w)(0.5) = 4 \quad \text{• Substitute in the volume formula.}$$
$$(w+3)(w) = 8 \quad \text{• Multiply each side by 2.}$$
$$w^2 + 3w = 8 \quad \text{• Simplify the left side.}$$
$$w^2 + 3w - 8 = 0 \quad \text{• Write in } ax^2 + bx + c = 0 \text{ form.}$$
$$w = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-8)}}{2(1)} \quad \text{• Use the quadratic formula.}$$
$$= \frac{-3 \pm \sqrt{41}}{2}$$
$$\approx 1.7 \quad \text{or} \quad -4.7$$

(continued)

Integrating Technology

In many application problems, it is helpful to use a calculator to estimate the solutions of a quadratic equation by applying the quadratic formula. For instance, the following figure shows the use of a graphing calculator to estimate the solutions of $w^2 + 3w - 8 = 0$.



We can disregard the negative value because the width must be positive. The width of the new candy bar, to the nearest tenth of an inch, should be 1.7 inches. The length should be 3 inches longer, which is 4.7 inches.

▶ Try Exercise 94, page 107

Quadratic equations are often used to determine the height (position) of an object that has been dropped or projected. For instance, the *position equation* $s = -16t^2 + v_0t + s_0$ can be used to estimate the height of a projected object near the surface of Earth at a given time t in seconds. In this equation, v_0 is the initial velocity of the object in feet per second and s_0 is the initial height of the object in feet.

EXAMPLE 9 Determine the Time of Descent

Rebelml/Dreamstime.com

A ball is thrown downward with an initial velocity of 5 feet per second from the Golden Gate Bridge, which is 220 feet above the water. How long will it take for the ball to hit the water? Round your answer to the nearest hundredth of a second.

Solution

The distance s, in feet, of the ball above the water after t seconds is given by the position equation $s = -16t^2 - 5t + 220$. We have replaced v_0 with -5 because the ball is thrown downward. (If the ball had been thrown upward, we would use $v_0 = 5$.) To determine the time it takes the ball to hit the water, substitute 0 for s in the equation $s = -16t^2 - 5t + 220$ and solve for t. In the following work, we have solved by using the quadratic formula.

$$0 = -16t^2 - 5t + 220$$

$$t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-16)(220)}}{2(-16)} \quad \text{• Use the quadratic formula.}$$

$$= \frac{5 \pm \sqrt{14{,}105}}{-32}$$

$$\approx -3.87 \quad \text{or} \quad 3.56 \quad \text{• Use a calculator to estimate } t.$$

Because the time must be positive, we disregard the negative value. The ball will hit the water in about 3.56 seconds.

▶ Try Exercise 96, page 107

EXERCISE SET 1.3

Concept Check

1. What does the zero product principle tell you about the equation $(x - 8)(x + 5) = 0$?

2. Simplify: $\sqrt{(x-3)^2}$

3. What do you add to the given expression to complete the square?

   a. $x^2 - 12x$ b. $x^2 + 9x$

4. What is the first step of solving the equation $4x^2 + 6x - 1 = 0$ by completing the square?

5. State the quadratic formula for the solutions of the equation $ax^2 + bx + c = 0$. Which part of the quadratic formula is called the discriminant?

6. The solutions of a quadratic equation are $-5 \pm \sqrt{2}$. Write the two solutions separately and find the value of each to the nearest tenth.

▢ Indicates Try It Exercises

In Exercises 7 to 16, solve each quadratic equation by factoring and applying the zero product principle.

7. $x^2 - 2x - 15 = 0$

8. $x^2 + 3x - 10 = 0$

9. $2x^2 - x = 1$

10. $2x^2 + 5x = 3$

11. $8x^2 + 189x - 72 = 0$

12. $12x^2 - 41x + 24 = 0$

13. $(x - 3)(x + 4) = 8$

14. $(2x + 1)(x - 3) = 9$

15. $3x^2 + x - 1 = (2x + 9)(x - 1)$

16. $(2x - 5)(x + 4) = (x + 4)(x - 2)$

In Exercises 17 to 34, use the square root procedure to solve the equation.

17. $y^2 = 24$

18. $y^2 = 48$

19. $z^2 = -16$

20. $z^2 = -100$

21. $(x - 5)^2 = 36$

22. $(x + 4)^2 = 121$

23. $(x + 2)^2 = 27$

24. $(x - 3)^2 = 8$

25. $(z - 4)^2 + 25 = 0$

26. $(z + 1)^2 + 64 = 0$

27. $(y - 6)^2 - 4 = 14$

28. $(y + 2)^2 + 5 = 15$

29. $5(x + 6)^2 + 60 = 0$

30. $(x + 2)^2 + 28 = 0$

31. $2(x + 4)^2 = 9$

32. $3(x - 2)^2 = 20$

33. $4(x - 2)^2 + 15 = 0$

34. $6(x + 5)^2 + 21 = 0$

In Exercises 35 to 50, solve each equation by completing the square.

35. $x^2 - 2x - 15 = 0$

36. $x^2 + 2x - 8 = 0$

37. $2x^2 - 5x - 12 = 0$

38. $3x^2 - 5x - 2 = 0$

39. $x^2 + 6x + 1 = 0$

40. $x^2 + 8x - 10 = 0$

41. $x^2 + 3x - 1 = 0$

42. $x^2 + 7x - 2 = 0$

43. $3x^2 - 8x = -1$

44. $2x^2 + 10x - 3 = 0$

45. $x^2 + 4x + 5 = 0$

46. $x^2 - 6x + 10 = 0$

47. $4x^2 + 4x + 2 = 0$

48. $9x^2 + 12x + 5 = 0$

49. $3x^2 + 2x + 1 = 0$

50. $4x^2 - 4x = -15$

In Exercises 51 to 70, solve by using the quadratic formula.

51. $x^2 - 2x = 15$

52. $x^2 - 5x = 24$

53. $12x^2 - 11x - 15 = 0$

54. $10x^2 + 19x - 15 = 0$

55. $x^2 - 2x = 2$

56. $x^2 + 4x - 1 = 0$

57. $x^2 = -x + 1$

58. $2x^2 + 4x = 1$

59. $4x^2 = 41 - 8x$

60. $2x = 9 - 3x^2$

61. $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$

62. $x^2 + \frac{5x}{2} - \frac{19}{8} = 0$

63. $x^2 + 6x + 13 = 0$

64. $x^2 = 2x - 26$

65. $2x^2 = 2x - 13$

66. $9x^2 - 24x + 20 = 0$

67. $x^2 + 2x + 29 = 0$

68. $x^2 + 6x + 21 = 0$

69. $4x^2 + 4x + 13 = 0$

70. $9x^2 = 12x - 49$

In Exercises 71 to 80, determine the discriminant of the quadratic equation and then state the number of real solutions of the equation. Do not solve the equation.

71. $2x^2 - 5x - 7 = 0$

72. $x^2 + 3x - 11 = 0$

73. $3x^2 - 2x + 10 = 0$

74. $x^2 + 3x + 3 = 0$

75. $x^2 - 20x + 100 = 0$

76. $4x^2 + 12x + 9 = 0$

77. $24x^2 = -10x + 21$

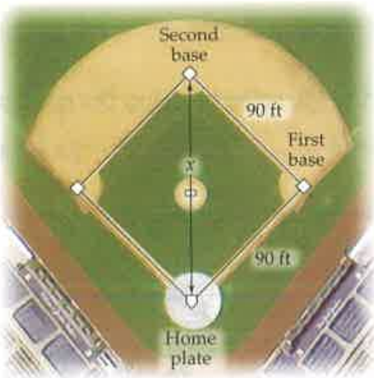
78. $32x^2 - 44x = -15$

79. $12x^2 + 15x = -7$

80. $8x^2 = 5x - 3$

81. **Suspension Bridge** The height h, in feet, that a suspension cable that supports a footbridge hangs above the bridge can be approximated by the equation $h = 0.045x^2 - 1.33x + 20$, where x is the distance, in feet, measured from the left side of the bridge. At what distance from the left side of the bridge is the height of the cable 11 feet above the bridge? Round to the nearest tenth of a foot.

82. **Dimensions of a Baseball Diamond** How far, to the nearest tenth of a foot, is it from home plate to second base on a baseball diamond? (*Hint:* The bases in a baseball diamond form a square that measures 90 feet on each side.)



83. **Volume of a Trough** Water is running into a trapezoidal trough. The volume V, in cubic feet, of water in the trough is given by $V = 32d^2 + 32d$, where d is the depth, in feet, of the water in the trough. Find the depth of the water when the volume is 850 cubic feet. Round to the nearest tenth of a foot.

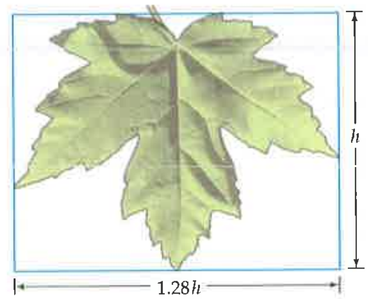
84. **Geometry** The length of each side of an equilateral triangle is 12 inches. Find the altitude of the triangle. Round to the nearest tenth of an inch.

85. **Sports** The height of a kicked football during a field goal attempt can be approximated by $h = -0.0114x^2 + 1.732x$, where h is the height, in feet, of the football when it is x feet from the kicker. To clear the goalpost, the football must be at least 10 feet above the ground. What is the maximum number of yards the kicker can be from the goalpost so that the football clears it? Round to the nearest tenth of a yard.

86. **Publishing Costs** The cost, in dollars, of publishing x books is $C(x) = 40{,}000 + 20x + 0.0001x^2$. How many books can be published for $250,000?

87. **Revenue** The demand for a certain product is given by $p = 26 - 0.01x$, where x is the number of units sold per month and p is the price, in dollars, at which each item is sold. The monthly revenue is given by $R = xp$. What number of items sold produces a monthly revenue of $16,500?

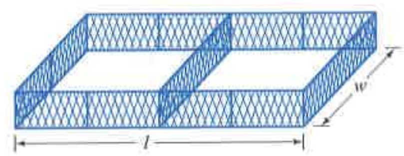
88. **Quadratic Growth** A plant's ability to create food through the process of photosynthesis depends on the surface area of its leaves. A biologist has determined that the surface area A of a maple leaf can be closely approximated by the formula $A = 0.72(1.28)h^2$, where h is the height of the leaf in inches.



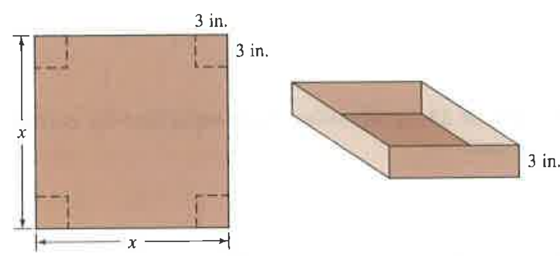
a. Find the surface area of a maple leaf with a height of 7 inches. Round to the nearest tenth of a square inch.

b. Find the height of a maple leaf with an area of 92 square inches. Round to the nearest tenth of an inch.

89. **Dimensions of an Animal Enclosure** A veterinarian wishes to use 132 feet of chain-link fencing to enclose a rectangular region and subdivide the region into two smaller rectangular regions, as shown in the following figure. If the total enclosed area is 576 square feet, find the dimensions of the enclosed region.



90. **Construction of a Box** A square piece of cardboard is formed into a box by cutting out 3-inch squares from each of the corners and folding up the sides, as shown in the following figure. If the volume of the box needs to be 126.75 cubic inches, what size square piece of cardboard is needed?

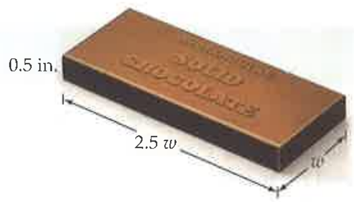


91. **Population Density of a City** The population density D (in people per square mile) of a city is related to the horizontal distance x, in miles, from the center of the city by the equation $D = -45x^2 + 190x + 200$, $0 < x < 5$. At what distances from the center of the city does the population density equal 250 people per square mile? Round each result to the nearest tenth of a mile.

92. **Traffic Control** Traffic engineers install "flow lights" at the entrances of freeways to control the number of cars entering the freeway during times of heavy traffic. For a particular freeway entrance, the number of cars N waiting to enter the freeway during the morning hours can be approximated by $N = -5t^2 + 80t - 280$, where t is the time of the day and $6 \le t \le 10.5$. According to this model, when will there be 35 cars waiting to enter the freeway?

93. **Daredevil Motorcycle Jump** In March 2000, Doug Danger made a successful motorcycle jump over an L-1011 jumbo jet. The horizontal distance of his jump was 160 feet, and his height, in feet, during the jump was approximated by $h = -16t^2 + 25.3t + 20$, $t \geq 0$. He left the takeoff ramp at a height of 20 feet, and he landed on the landing ramp at a height of about 17 feet. How long, to the nearest tenth of a second, was he in the air?

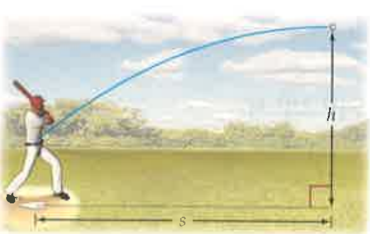
94. **Dimensions of a Candy Bar** A company makes rectangular solid candy bars that measure 5 inches by 2 inches by 0.5 inch. Due to difficult financial times, the company has decided to keep the price of the candy bar fixed and reduce the volume of the bar by 20%. What should be the dimensions, to the nearest tenth of an inch, of the new candy bar if the company decides to keep the height at 0.5 inch and to make the length of the new candy bar 2.5 times as long as its width?



95. **Height of a Rocket** A model rocket is launched upward with an initial velocity of 220 feet per second. The height, in feet, of the rocket t seconds after the launch is given by the equation $h = -16t^2 + 220t$. How many seconds after the launch will the rocket be 350 feet above the ground? Round to the nearest tenth of a second.

96. **Baseball** The height h, in feet, of a baseball above the ground t seconds after it is hit is given by $h = -16t^2 + 52t + 4.5$. Use this equation to determine the number of seconds, to the nearest tenth of a second, from the time the ball is hit until the ball touches the ground.

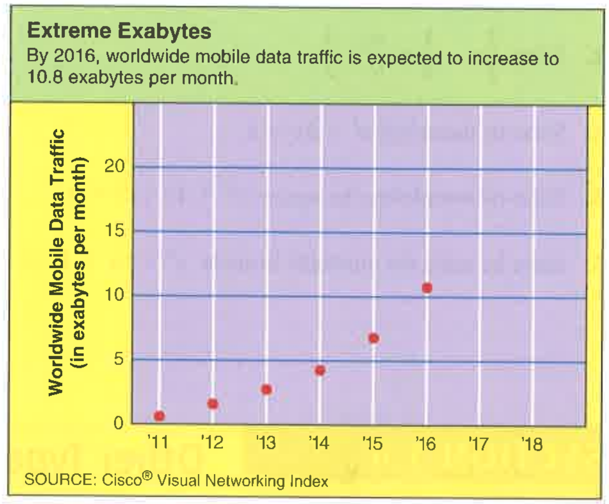
97. **Baseball** Two equations can be used to track the position of a baseball t seconds after it is hit. For instance, suppose $h = -16t^2 + 50t + 4.5$ gives the height, in feet, of a baseball t seconds after it is hit and $s = 103.9t$ gives the horizontal distance, in feet, of the ball from home plate t seconds after it is hit. (See the following figure.) Use these equations to determine whether this particular baseball will clear a 10-foot fence positioned 360 feet from home plate.



98. **Basketball** Michael Jordan was known for his hang time, which is the amount of time a player is in the air when making a jump toward the basket. An equation that approximates the height h, in inches, of one of Jordan's jumps is given by $h = -16t^2 + 26.6t$, where t is time in seconds. Use this equation to determine Michael Jordan's hang time, to the nearest tenth of a second, for this jump.

99. **Orbital Debris** The amount of space debris is increasing. The total mass M, in millions of kilograms, of objects in Earth orbit can be modeled by $M = 0.0001t^2 + 0.16t + 4.34$, where $t = 0$ corresponds to the year 2000. (*Source:* http://orbitaldebris.jsc.nasa.gov/) According to this model, in what year will the total mass of space debris objects first exceed 10 million kilograms?

100. **Mobile Data** The projected worldwide mobile data traffic D, in exabytes per month, can be modeled by $D = 0.40x^2 - 8.81x + 49.25$, $11 \leq x \leq 16$, where $x = 11$ corresponds to the year 2011. (*Source:* www.cisco.com) Suppose this trend continues beyond the year 2016. Use the model to predict the first year in which worldwide mobile data traffic will always be greater than 25 exabytes per month. *Note:* 1 exabyte $= 10^{18}$ bytes, or 1 billion gigabytes.



101. **Centenarians** According to data provided by the U.S. Census Bureau, the number N, in thousands, of centenarians (a person whose age is 100 years or older) who will be living in the United States during a year from 2010 to 2050 can be approximated by $N = 0.3453x^2 - 9.417x + 164.1$, where x is the number of years after the beginning of 2000. Use this equation to determine in what year there be 200,000 centenarians living in the United States.

102. **Automotive Engineering** The number N of feet that a car needs to stop on a certain road surface is given by the equation $N = -0.015v^2 + 3v$, $0 \leq v \leq 90$, where v is the speed of the car in miles per hour when the driver applies the brakes. What is the maximum speed, to the nearest mile per hour, that a motorist can be traveling and stop the car within 100 feet?

Enrichment Exercises

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ was given as $b^2 - 4ac$. We will call this the *coefficient* definition. There is, however, a more general definition of the discriminant that can be extended to polynomial equations of any degree. Let r_1 and r_2 be the solutions of the quadratic equation $ax^2 + bx + c = 0$. Then the discriminant of the equation is given by $a^2(r_1 - r_2)^2$. That is, the discriminant is the product of the square of the leading coefficient and the square of the difference between the roots. We will call this the *roots* definition of the discriminant.

103. Verify that the coefficient definition and the roots definition give the same value for the discriminant of each of the following equations.

a. $x^2 - x - 6 = 0$

b. $9x^2 - 6x - 1 = 0$

c. $x^2 + 4x + 4 = 0$

104. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $r_1 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$. Show that $a^2(r_1 - r_2)^2 = b^2 - 4ac$.

105. Let r_1, r_2, and r_3 be the solutions of the cubic equation $ax^3 + bx^2 + cx + d = 0$. Then the discriminant of the cubic equation is $a^4(r_1 - r_2)^2(r_1 - r_3)^2(r_2 - r_3)^2$. Find the discriminant of $x^3 - 4x^2 - 4x + 16 = 0$.

106. Suppose the discriminant of a cubic equation is 0. What can be said about the roots of that equation?

MID-CHAPTER 1 QUIZ

1. Solve: $6 - 4(2x + 1) = 5(3 - 2x)$

2. Solve: $\dfrac{2}{3}x - \dfrac{1}{4} = \dfrac{x}{6} + \dfrac{3}{2}$

3. Solve by factoring: $x^2 - 5x = 6$

4. Solve by completing the square: $x^2 + 4x - 2 = 0$

5. Solve by using the quadratic formula: $x^2 - 6x + 12 = 0$

6. A runner runs a course at a constant speed of 8 miles per hour. One hour later, a cyclist begins the same course at a constant speed of 16 miles per hour. How long after the runner starts does the cyclist overtake the runner?

7. A pharmacist mixes a 9% acetic acid solution with a 4% acetic solution. How many milliliters of each solution should the pharmacist use to make a 500-milliliter solution that is 6% acetic acid?

8. A mason can complete a wall in 10 hours, but an apprentice mason requires 15 hours to do the same job. How long will it take to build the wall with both people working?

SECTION 1.4 Other Types of Equations

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A5.

PS1. Factor: $x^3 - 16x$ [P.4]

PS2. Factor: $x^4 - 36x^2$ [P.4]

PS3. Evaluate: $8^{2/3}$ [P.2]

PS4. Evaluate: $16^{3/2}$ [P.2]

PS5. Multiply: $(1 + \sqrt{x - 5})^2$, $x > 5$ [P.2/P.3]

PS6. Multiply: $(2 - \sqrt{x + 3})^2$, $x > -3$ [P.2/P.3]

Polynomial Equations

Some polynomial equations that are neither linear nor quadratic can be solved by the various techniques presented in this section. For instance, the **third-degree equation,** or **cubic equation,** in Example 1 can be solved by factoring the polynomial and using the zero product principle.